The Art of PD Curve Calibration

Dirk Tasche

Bank of England – Prudential Regulation Authority

dirk.tasche@gmx.net

Credit Scoring and Credit Control XIII
Edinburgh
August 2013

The opinions expressed in this presentation are those of the author and do not necessarily reflect views of the Bank of England.
Outline

Background

Estimation framework

PD calibration based on the likelihood ratio

Conclusions

References
Outline

Background

Estimation framework

PD calibration based on the likelihood ratio

Conclusions

References
Data

S&P rating frequencies (%) and default rates (%) in 2009 and rating frequencies in 2010\(^2\).

<table>
<thead>
<tr>
<th>Rating</th>
<th>2009 Freq</th>
<th>DR</th>
<th>2010 Freq</th>
<th>DR</th>
<th>Rating</th>
<th>2009 Freq</th>
<th>DR</th>
<th>2010 Freq</th>
<th>DR</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>1.38</td>
<td>0</td>
<td>1.3</td>
<td>?</td>
<td>BBB-</td>
<td>7.83</td>
<td>1.09</td>
<td>7.79</td>
<td>?</td>
</tr>
<tr>
<td>AA+</td>
<td>0.63</td>
<td>0</td>
<td>0.45</td>
<td>?</td>
<td>BB+</td>
<td>4.54</td>
<td>0</td>
<td>4.6</td>
<td>?</td>
</tr>
<tr>
<td>AA</td>
<td>3.21</td>
<td>0</td>
<td>2.59</td>
<td>?</td>
<td>BB</td>
<td>5.03</td>
<td>1.02</td>
<td>5</td>
<td>?</td>
</tr>
<tr>
<td>AA-</td>
<td>4.18</td>
<td>0</td>
<td>3.78</td>
<td>?</td>
<td>BB-</td>
<td>7.53</td>
<td>0.91</td>
<td>6.86</td>
<td>?</td>
</tr>
<tr>
<td>A+</td>
<td>5.8</td>
<td>0.29</td>
<td>6.39</td>
<td>?</td>
<td>B+</td>
<td>7.47</td>
<td>5.48</td>
<td>7.12</td>
<td>?</td>
</tr>
<tr>
<td>A</td>
<td>8.7</td>
<td>0.39</td>
<td>8.58</td>
<td>?</td>
<td>B</td>
<td>8.23</td>
<td>9.96</td>
<td>7.9</td>
<td>?</td>
</tr>
<tr>
<td>A-</td>
<td>9.32</td>
<td>0</td>
<td>9.56</td>
<td>?</td>
<td>B-</td>
<td>5.17</td>
<td>17.16</td>
<td>5.25</td>
<td>?</td>
</tr>
<tr>
<td>BBB+</td>
<td>8.5</td>
<td>0.4</td>
<td>8.28</td>
<td>?</td>
<td>CCC-C</td>
<td>3.24</td>
<td>48.42</td>
<td>3.98</td>
<td>?</td>
</tr>
<tr>
<td>BBB</td>
<td>9.23</td>
<td>0.18</td>
<td>10.56</td>
<td>?</td>
<td>All</td>
<td>100</td>
<td>3.99</td>
<td>100</td>
<td>1.14</td>
</tr>
</tbody>
</table>

\(^2\)Sources: [S&P(2010)], tables 51 to 53, and [S&P(2011)], tables 50 to 52
Problem

- **Forecast grade-level default rates** for 2010, based on
  - rating frequencies and grade-level default rates observed in 2009, and
  - rating frequencies observed at the beginning of 2010.
- Consider two cases:
  1. Overall default rate for 2010 is not known (hence to be forecast).
  2. An independent forecast of the overall default rate for 2010 is available.
- Economically motivated constraints for forecasts:
  - Forecast default rates may be small but must be positive.
  - Forecast default rates must strictly increase with lower creditworthiness.
Observations from slide 4

- Rating frequencies in 2009 and 2010 are statistically significantly different.
- The overall default rate cannot be considered constant.
- The empirical default rates can be zero.
- The empirical default rates need not be monotonous.
- **Consequence:** The default rates observed in 2009 are not likely to be good forecasts of the default rates in 2010.
Outline

Background

Estimation framework

PD calibration based on the likelihood ratio

Conclusions

References
Estimation framework

One-period model

- Pair of random variables \((X, S)\):
  - Rating grade at beginning of period \(X \in \{1, 2, \ldots, k\}\)
  - \(X = 1\) means low creditworthiness
  - Solvency state at end of period \(S \in \{0, 1\}\)
  - \(\{S = 1\} = D\) ‘default’, \(\{S = 0\} = N\) ‘survival’
- Marginal distributions of \(X\) and \(S\):
  - \(x \mapsto P[X = x]\) ‘rating profile’
  - \(p = P[D] = 1 - P[N]\) ‘unconditional probability of default’ (PD)
- Conditional marginal distributions:
  - \(x \mapsto P[D \mid X = x]\) ‘PD curve’
  - \(x \mapsto P[X = x \mid D], x \mapsto P[X = x \mid N]\) ‘conditional rating profiles’
Problems in mathematical terms

- Subscript 0 for quantities related to 2009, subscript 1 for quantities related to 2010

- **Problem I.** Observed PD curve \( x \mapsto P_0[D \mid X = x] \) not positive and not monotonous \( \Rightarrow \) Fit ‘smoothed’ PD curve.

- **Problem II.** Which model components from 2009 can be assumed invariant and re-used for 2010?

- **Problem III.** How to compare performance of solution approaches?

- Solution for I: ‘Quasi moment matching’ – see [Tasche(2012)].
Invariant or not?

- Rating profile $x \mapsto \Pr[X = x]$ is not invariant (as empirically observed)
- Unconditional PD $p$ is not invariant (as empirically observed)
- **PD curve** $x \mapsto \Pr[D \mid X = x]$ is not invariant:
  - As empirically observed
  - Follows from non-invariance of $p$ because

\[
\Pr[D \mid X = x] = \frac{p \Pr[X = x \mid D]}{p \Pr[X = x \mid D] + (1 - p) \Pr[X = x \mid \neg D]} 
\]  

(1)

- Can $x \mapsto \Pr[X = x \mid D]$ and $x \mapsto \Pr[X = x \mid \neg D]$ be invariant at the same time? Unlikely – see [Tasche(2012)].
Weaker invariance assumptions

1. Default profile $x \mapsto P[X = x \mid D]$ is invariant but survival profile $x \mapsto P[X = x \mid N]$ is not.

2. **Likelihood ratio** $x \mapsto \lambda(x) = \frac{P[X = x \mid N]}{P[X = x \mid D]}$ is invariant.

3. Discriminatory power (accuracy ratio) is invariant:

   $$AR = \sum_{x=2}^{k} P[X = x \mid N] P[X \leq x - 1 \mid D]$$
   $$- \sum_{x=1}^{k-1} P[X = x \mid N] P[X \geq x + 1 \mid D]$$  \hspace{1cm} (2a)

4. Scaled PD curve: There is a constant $c_{PD}$ such that

   $$P_1[D \mid X = x] = c_{PD} P_0[D \mid X = x], \quad x = 1, \ldots, k.$$  \hspace{1cm} (2b)

5. **Scaled likelihood ratio:** There is a constant $c_{LR} > 0$ such that

   $$\lambda_1(x) = c_{LR} \lambda_0(x), \quad x = 1, \ldots, k.$$  \hspace{1cm} (2c)
Outline

Background

Estimation framework

PD calibration based on the likelihood ratio

Conclusions

References
Properties of the likelihood ratio

- Neyman & Pearson lemma $\Rightarrow \lambda(X)$ is the most powerful statistic for testing ‘default’ against ‘survival’.
- PD curve and likelihood ratio are closely related:
  \[ \Pr[D \mid X = x] = \frac{p}{p + (1 - p) \lambda(x)}, \quad x = 1, \ldots, k. \quad (3a) \]
- Default profile and likelihood ratio are closely related:
  \[ \Pr[X = x \mid D] = \frac{\Pr[X = x]}{p + (1 - p) \lambda(x)}, \quad x = 1, \ldots, k. \quad (3b) \]
- $(3b) \Rightarrow \text{Unconditional PD } p \text{ is uniquely determined by rating profile and likelihood ratio.}$
Likelihood ratio and unconditional PD

Proposition 1

Let \( \pi_x > 0, \ x = 1, \ldots, k \) be a probability distribution. Assume that \( x \mapsto \lambda(x) > 0 \) is non-constant for \( x = 1, \ldots, k \). Then

\[
\sum_{x=1}^{k} \frac{\pi_x}{p + (1 - p) \lambda(x)} = 1
\]  \hspace{1cm} (4a)

has a unique solution \( 0 \leq p < 1 \) if and only it holds that

\[
\sum_{x=1}^{k} \frac{\pi_x}{\lambda(x)} \geq 1 \quad \text{and} \quad \sum_{x=1}^{k} \pi_x \lambda(x) > 1. \]  \hspace{1cm} (4b)

Proof. \( f(p) = \sum_{x=1}^{k} \frac{\pi_x}{p + (1 - p) \lambda(x)} \). Three cases:

A: \( \sum \frac{\pi}{\lambda} > 1 \) and \( \sum \pi \lambda \leq 1 \),

B: \( \sum \frac{\pi}{\lambda} \geq 1 \) and \( \sum \pi \lambda > 1 \),

C: \( \sum \frac{\pi}{\lambda} < 1 \) and \( \sum \pi \lambda > 1 \).
Illustration of the proof of Proposition 1

\begin{center}
\begin{tikzpicture}
\begin{axis}[
    width=\textwidth,
    xlabel={p},
    ylabel={f(p)},
    xmin=0, xmax=1,
    ymin=0, ymax=2,
]
\addplot[\dashed, thick, black] coordinates {
(0,2) (1,0)
};
\addplot[\dashed, thick, black] coordinates {
(0,0) (1,1)
};
\addplot[\dashed, thick, black] coordinates {
(0,1) (1,0)
};
\legend{Case A, Case B, Case C}
\end{axis}
\end{tikzpicture}
\end{center}
Consequences of Proposition 1

- Solve (4a) with data from slide 4:
  - $\pi_x = \text{rating profile of 2010}$
  - $\lambda(x) = (\text{smoothed}) \text{ likelihood ratio observed in 2009}$
- Resulting **forecast of 2010 unconditional default rate**: 5.38%.
- Observed 2010 unconditional default rate: 1.14%.
  - $\Rightarrow$ Likelihood ratio cannot be assumed to be invariant.
- Improved approach:
  - Independent estimate of 2010 unconditional default rate
  - Scaled 2009 likelihood ratio
Scaled likelihood ratio

**Conclusion from Proposition 1:** Let \( \pi_x > 0, \ x = 1, \ldots, k \) be a probability distribution. Assume that \( x \mapsto \lambda(x) > 0 \) is non-constant for \( x = 1, \ldots, k \). Let \( p \in (0, 1) \) be fixed. Then there is a unique number \( c_{LR} \) with

\[
\left( \sum_{x=1}^{k} \pi_x \lambda(x) \right)^{-1} < c_{LR} < \sum_{x=1}^{k} \frac{\pi_x}{\lambda(x)} \tag{5a}
\]

such that

\[
1 = \sum_{x=1}^{k} \frac{\pi_x}{p + (1-p) c_{LR} \lambda(x)} \tag{5b}
\]

**Solve (5b) with data from slide 4:**

- \( \pi_x \) = rating profile of 2010
- \( \lambda(x) \) = (smoothed) likelihood ratio observed in 2009
- \( p \) = forecast of 2010 unconditional default rate

**Compare 2010 grade-level default rates with forecasts based on approaches 4 and 5 of slide 11.**
Grade-level default rate forecasts for 2010

Observed 2010 default rates (%) vs. ‘scaled PDs’ and ‘scaled LR’ forecasts (%).

<table>
<thead>
<tr>
<th>Rating</th>
<th>Obs. DR</th>
<th>Sc. PD</th>
<th>Sc. LR</th>
<th>Rating</th>
<th>Obs. DR</th>
<th>Sc. PD</th>
<th>Sc. LR</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0</td>
<td>0.0007</td>
<td>0.0005</td>
<td>BBB-</td>
<td>0</td>
<td>0.2107</td>
<td>0.1581</td>
</tr>
<tr>
<td>AA+</td>
<td>0</td>
<td>0.0015</td>
<td>0.0011</td>
<td>BB+</td>
<td>0.7874</td>
<td>0.3006</td>
<td>0.2263</td>
</tr>
<tr>
<td>AA</td>
<td>0</td>
<td>0.0031</td>
<td>0.0023</td>
<td>BB</td>
<td>0.3623</td>
<td>0.4012</td>
<td>0.3029</td>
</tr>
<tr>
<td>AA-</td>
<td>0</td>
<td>0.0066</td>
<td>0.0049</td>
<td>BB-</td>
<td>0.5277</td>
<td>0.6024</td>
<td>0.4576</td>
</tr>
<tr>
<td>A+</td>
<td>0</td>
<td>0.0125</td>
<td>0.0093</td>
<td>B+</td>
<td>0</td>
<td>1.0417</td>
<td>0.8023</td>
</tr>
<tr>
<td>A</td>
<td>0</td>
<td>0.0241</td>
<td>0.018</td>
<td>B</td>
<td>0.6881</td>
<td>2.1134</td>
<td>1.6844</td>
</tr>
<tr>
<td>A-</td>
<td>0</td>
<td>0.0458</td>
<td>0.0342</td>
<td>B-</td>
<td>2.069</td>
<td>5.1671</td>
<td>4.5716</td>
</tr>
<tr>
<td>BBB+</td>
<td>0</td>
<td>0.0789</td>
<td>0.059</td>
<td>CCC-C</td>
<td>22.2727</td>
<td>12.7755</td>
<td>15.576</td>
</tr>
<tr>
<td>BBB</td>
<td>0</td>
<td>0.1307</td>
<td>0.0979</td>
<td>All</td>
<td>1.141</td>
<td>1.141</td>
<td>1.141</td>
</tr>
</tbody>
</table>

$\chi^2$-tests of implied default profiles against observed default numbers:

- Scaled PDs: p-value 4.1%
- Scaled LR: p-value 10.5%
Conclusions

Outline

Background

Estimation framework

PD calibration based on the likelihood ratio

Conclusions

References
Observations and conclusions

- Empirical grade-level default rates:
  - Strong variation over time
  - Often zero
  - Not monotonous with regard to creditworthiness
- Therefore, making positive and monotonous default rate forecasts is challenging.
- Compared ‘scaled PD curve’ and ‘scaled likelihood ratio’ approaches:
  - Scaled LR always gives a valid PD curve.
  - Scaled LR is not ‘contaminated’ by unconditional default rate of previous year (eq. (3a)).
  - Scaled LR gives better fit of observed default rates.
Outline

Background

Estimation framework

PD calibration based on the likelihood ratio

Conclusions

References
S&P.
Default, Transition, and Recovery: 2009 Annual Global Corporate Default Study And Rating Transitions.
Report, Standard & Poor’s, March 2010.

S&P.
Default, Transition, and Recovery: 2010 Annual Global Corporate Default Study And Rating Transitions.
Report, Standard & Poor’s, March 2011.

D. Tasche.
The art of PD curve calibration.