Fitting a distribution to Value-at-Risk and Expected Shortfall, with an application to covered bonds

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1The opinions expressed in this presentation are those of the author and do not necessarily reflect views of the Bank of England.
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The problem

- Covered bonds are an important funding instruments for many banks.
- They are considered very safe investments.
- **No loss** due to missed payments to bondholders was ever observed.
- **No historical data** on covered bonds defaults and related losses are available.
- Estimating expected loss for covered bonds is not straightforward.
Essential features of covered bonds

- The bond is issued by – or bondholders otherwise have full recourse to – a credit institution which is subject to public supervision and regulation;
- Bondholders have a claim against a cover pool of financial assets in priority to the unsecured creditors of the credit institution;
- The credit institution has the ongoing obligation to maintain sufficient assets in the cover pool to satisfy the claims of covered bondholders at all times;
- The obligations of the credit institution in respect of the cover pool are supervised by public or other independent bodies.

Source: The European Covered Bond Council (http://ecbc.hypo.org)
A structural model for covered bonds

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Two assets approach

- **Bond issuer’s asset values (random variables):**
  - $X$ is value of assets in cover pool (collateral for covered bonds).
  - $Y$ is value of issuer’s other assets.

- **Bond issuer’s debts (constants):**
  - $C$ is nominal value (principal) of covered bonds.
  - $S$ is nominal value of senior unsecured debt.
  - $U$ is nominal value of subordinated unsecured debt.

- **Double recourse:** In case of issuer’s default,
  - covered bonds are served by proceeds from cover pool;
  - if cover pool proceeds are insufficient bond holders have a claim against the issuer’s other assets, ranking pari passu with the holders of senior unsecured debt.

- $L_C$, $L_S$, $L_U$ denote **loss rates** for covered bonds, senior unsecured debt and subordinated debt respectively.
A structural model for covered bonds

Three loss events

▸ Issuer defaults, total assets sufficient for senior debt:
\[ C + S \leq X + Y < C + S + U \]

\[ \Rightarrow \quad L_U = 1 - \frac{X + Y - (C + S)}{U}, \quad L_C = L_S = 0. \quad (1a) \]

▸ Issuer defaults, total assets insufficient for senior debt, cover pool sufficient for covered bonds:
\[ X + Y < C + S, \quad X \geq C \]

\[ \Rightarrow \quad L_C = 0, \quad L_S = 1 - \frac{X + Y - C}{S}, \quad L_U = 1. \quad (1b) \]

▸ Issuer defaults, total assets insufficient for senior debt, cover pool insufficient for covered bonds:
\[ X + Y < C + S, \quad X < C \]

\[ \Rightarrow \quad L_C = \frac{(C - X)(S + C - X - Y)}{(S + C - X)C}, \quad L_S = \frac{S + C - X - Y}{S + C - X}, \quad L_U = 1. \quad (1c) \]
Observations

- Covered bonds holders only suffer loss if
  - the issuer defaults \((X + Y < C + S + U)\) and
  - the total assets value falls below the amount of senior debt \((X + Y < C + S)\) and
  - the value of the cover pool assets falls below the nominal value of the bonds \((X < C)\).

- Issuer’s PD (probability of default) \(P[X + Y < C + S + U]\) is bounded from above by

\[
P[X + Y < C + S + U] \leq P[X < C + S + U]. \tag{2}
\]

- The bound does not depend on the distribution of \(Y\) (value of issuer’s other assets).
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Specifying the asset distributions

- Lognormal distribution is the most convenient choice of an asset value distribution.

- **Assumptions:**
  - Both the cover pool value $X$ and the value of the issuer’s other assets $Y$ are lognormally distributed.
  - $X$ and $Y$ are linked by a normal copula.

- **Parametrisation:**

\[
X = \exp(\mu + \sigma \xi), \quad Y = \exp(\nu + \tau \eta), \quad (3)
\]

with $\mu, \nu \in \mathbb{R}$, $\sigma, \tau > 0$ and $(\xi, \eta) \sim \mathcal{N}\left((0, 0), \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$ for some $\rho \in [0, 1]$.

- **Drawback:** $X + Y$ is **not** lognormal.
Derive formulae by conditioning on \( X \) or \( Y \).

Case \( \varrho < 1 \):

\[
CE[L_C] = \int_{-\infty}^{\log(C) - \mu / \sigma} \left( C - e^{\mu + \sigma X} \right) \varphi(X) \Phi \left( \frac{\log(C + S - e^{\mu + \sigma X}) - (\nu + \tau \varrho X)}{\tau \sqrt{1 - \varrho^2}} \right) dx
\]

\[
- e^{\nu + \tau^2 (1 - \varrho^2) / 2} \int_{-\infty}^{\log(C) - \mu / \sigma} \frac{\left( C - e^{\mu + \sigma X} \right) e^{\tau \varrho X}}{C + S - e^{\mu + \sigma X}} \varphi(X) \Phi \left( \frac{\log(C + S - e^{\mu + \sigma X}) - (\nu + \tau \varrho X)}{\tau \sqrt{1 - \varrho^2}} - \tau \sqrt{1 - \varrho^2} \right) dx. \tag{4}
\]
Formulae for covered bonds expected loss II

- Consider also **strongest dependence** between $X$ and $Y$, i.e. comonotonicity.

- Comonotonic case, $\varrho = 1$:

$$
CE[L_C] = \min \left( \frac{\log(C) - \mu}{\sigma}, x(C+S) \right) \int_{-\infty}^{\varphi(x)} \frac{(C - e^{\mu + \sigma x})(C+S - e^{\mu + \sigma x} - e^{\nu + \tau x})}{C+S - e^{\mu + \sigma x}} \, dx.
$$

(5a)

- For $a > 0$, $x(a)$ denotes the unique solution of

$$
a = e^{\mu + \sigma x} + e^{\nu + \tau x}.
$$

(5b)
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The comonotonic case

- So far, a complete solution is known only in the comonotonic case.
- **Assumption:** We know
  - the issuer’s $PD_{\text{issuer}}$ and expected loss $EL_{\text{issuer}}$, and
  - the $PD_{\text{cover}}$ and $EL_{\text{cover}}$ of the cover pool.
- Fitting the distribution (3) of $X$ to $PD_{\text{cover}}$ and $EL_{\text{cover}}$ is equivalent to fitting a lognormal distribution to given Value-at-Risk and Expected Shortfall.
- A unique lognormal distribution for $X$ can be fitted as long as we have $0 < EL_{\text{cover}} < PD_{\text{cover}} < 1$. 
Fitting the cover pool asset value distribution

- Denote by \( \nu \) the **level of over-collateralisation** of the covered bonds.
- First, determine \( \sigma \) in (3) for the distribution of \( X \) by solving
  
  \[
  0 = \Phi(\Phi^{-1}(PD_{cover}) - \sigma) - (PD_{cover} - EL_{cover}) \exp(\sigma \Phi^{-1}(PD_{cover}) - \sigma^2/2).
  \]
  (6a)

- Then calculate \( \mu \):
  
  \[
  \mu = \log((1 + \nu) C) - \sigma \Phi^{-1}(PD_{cover}),
  \]
  (6b)

- Factor \( 1 + \nu \) reflects the fact that PD and EL refer to the entire pool including the assets for over-collateralisation.
Fitting the distribution of the value of other assets I

In the comonotonic case, we need to solve this equation system for parameters $\nu$ and $\tau$ for $Y$ (see (5b) for $x(a)$):

$$PD_{issuer} = \Phi(x(C + S + U)),$$  \hspace{1cm} (7a)

$$(C + S + U) \cdot PD_{issuer} (1 - LGD_{issuer})$$

$$= e^{\mu + \sigma^2/2} \Phi(x(C + S + U) - \sigma)$$

$$+ e^{\nu + \tau^2/2} \Phi(x(C + S + U) - \tau).$$  \hspace{1cm} (7b)

Due to (2) and other dependence issues, there is not always a solution $(\nu, \tau)$ of (7a), (7b).
Proposition. Assume that $\mu \in \mathbb{R}$ and $\sigma > 0$ are fixed. Then there is a solution $(\nu, \tau)$ of (7a), (7b) if and only if

$$0 < PD_{\text{issuer}} < \Phi \left( \frac{\log(C+S+U)-\mu}{\sigma} \right) \quad \text{and} \quad (8a)$$

$$PD_{\text{issuer}} \frac{e^{\mu+\sigma} \Phi^{-1}(PD_{\text{issuer}}) - e^{\mu+\sigma^2/2} \Phi\left(\Phi^{-1}(PD_{\text{issuer}}) - \sigma\right)}{PD_{\text{issuer}} (C+S+U)} < LGD_{\text{issuer}}$$

$$< 1 - \frac{e^{\mu+\sigma^2/2} \Phi\left(\Phi^{-1}(PD_{\text{issuer}}) - \sigma\right)}{PD_{\text{issuer}} (C+S+U)}. \quad (8b)$$

If there is a solution $(\nu, \tau)$ of (7a), (7b) it is unique.

Next slide: Illustration for $PD_{\text{cover}} = 0.05\%$, $LGD_{\text{cover}} = 30\%$ (small range) and $PD_{\text{cover}} = 0.5\%$, $LGD_{\text{cover}} = 50\%$ (large range).
Illustration of feasible PD and LGD range for issuer
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We have presented an approach to the calculation of expected loss for covered bonds, based on a structural approach with two asset values.

A number of open issues remains to be solved:

- How to estimate the asset correlation?
- How to define ’default’ of the cover pool? As ’losses exceed loss reserve’?
- How to better reflect different horizons (one year for issuer’s default, – say – ten years covered bonds maturity)?
- What to do in the case where there is no solution for a combination of issuer’s PD and LGD as well as cover pool PD and LGD?
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