Default and prepayment: an NPV analysis under a Markovian dynamics of the credit market

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Aim of the presentation

How to evaluate the customer’s credit relation when the payment process depends on changing credit market conditions (CMCs)

The main topic is to analyze the Net Present Value (NPV) taking into account default risk, prepayment risk and CMCs for a single transaction that is representative of a given population of customers, for example accepted and rejected ones.

The model will be generalized at a portfolio level, showing how to evaluate the first two moments of its NPV.

The application of the model is valuable to accept or reject loans in the credit granting process.
Steps of the analysis

Input 1: Credit market conditions
(given by a Markov Chain)

Input 2: Customer’s Payment Process
(default and prepayment random times)

Output: NPV as a random variable
The CMCs may be modeled by a homogeneous Markov Chain

Over the loan term, $n$, at every period the CMCs take a distinct state in a given finite set; the time period can be assumed varying on a monthly base.

For instance, with two states $G$ (Good status) and $B$ (Bad one), a possible scenario, given by a sequence of states, is

$$(B, B, G, ..., B, ..., G, G)$$

States change according to a homogeneous Markov Chain.

Example: Initial state $B$ with transition probabilities matrix

<table>
<thead>
<tr>
<th></th>
<th>CMC Bad</th>
<th>CMC Good</th>
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<tbody>
<tr>
<td>CMC Bad</td>
<td>0.92</td>
<td>0.08</td>
</tr>
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<td>CMC Good</td>
<td>0.04</td>
<td>0.96</td>
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The local probabilities of default and prepayment depend on the CMCs and represent the second input for the model

Input 2

The local probabilities of default and prepayment are for \( i = 1, \ldots, n \)

\[ \lambda_i \in \{\lambda_i(B),\lambda_i(G)\} \quad \mu_i \in \{\mu_i(B),\mu_i(G)\} \]

The probability distributions of the default time and prepayment time are

\[ d_1 = \lambda_1 \quad e_1 = \mu_1 \]

\[ d_h = \prod_{i=1}^{h-1} (1 - \lambda_i - \mu_i) \cdot \lambda_h \quad e_h = \prod_{i=1}^{h-1} (1 - \lambda_i - \mu_i) \cdot \mu_h \]

for \( h = 2, \ldots, n \) for \( h = 2, \ldots, n-1 \)
Other parameters as inputs for the model

The other parameters of the model

• Loan information: granted amount $f$, term $n$, instalments $r_i (i = 1, \ldots, n)$
• Deterministic recovery rate in case of default: $t_i (i = 1, \ldots, n)$
• Evaluation rate and cost of funding: $y$ constant over the term of the loan
NPV is a mixture random variable

The NPV variable is a mixture of the conditional ones in which the conditioning events are all the possible scenarios \(2^n\)

In order to manage the huge number of possible scenarios, an approximation of their distribution can be obtained by simulation

For any given scenario \(H\), the evaluation of the first two moments of the conditional NPV, \(E(NPV^k | H)\), \(k = 1, 2\), is done
The relation that links the $k$-th moment of the conditional NPV to the inputs show the dependency on all the risk parameters.

**First two moments of NPV conditioned to a given scenario $H$**

$$E[NPV^k|H] = P(E) \cdot \left( \sum_{j=1}^{n} r_j \cdot (1+y)^{-j} - f \right)^k$$

$$+ \sum_{h=1}^{n} d_h \cdot \left( \sum_{j=1}^{h-1} r_j \cdot (1+y)^{-j} + t_h \cdot (1+y)^{-h} \cdot \left( \sum_{j=h}^{n} r_j \cdot (1+x)^{-(j-h)} \right) - f \right)^k$$

$$+ \sum_{h=1}^{n-1} e_h \cdot \left( \sum_{j=1}^{h-1} r_j \cdot (1+y)^{-j} + (1+y)^{-h} \cdot \left( \sum_{j=h}^{n} r_j \cdot (1+x)^{-(j-h)} \right) - f \right)^k$$

$$P(E) = 1 - \sum_{h=1}^{n} d_h - \sum_{h=1}^{n-1} e_h$$

$k = 1, 2$

$x$ is the contractual internal rate of the operation
In the model development several choices have to be made

Some practitioner’s tips

1) The Markov Chain and the initial probability distribution of the states can be estimated, for example, by a Hidden Markov Model using firm’s own data. This approach can be used if one supposes that the CMCs can be interpreted as latent variables

2) The local probabilities of default and prepayment have to be estimated with reference to the entire population (i.e. considering rejected applicants also); if one is interested in applying any reject inference technique, a Heckman model type is summarized in the appendix of this presentation where it is shown how it works for a continuous outcome variable (see Conference Papers)

3) Constant local probabilities of default and prepayment depending on the CMCs can be used as a first approximation
For a given loan, two different simulations are performed depending on the probability distribution over the initial state

An example

- \( f = €8,500, n = 60 \) months, \( r_i = 190 \) euros
- Constant recovery rate, \( t_i = 30\% \)
- \( y = 500 \) bps (yearly base)
- Constant \( \lambda_i \) in \{0.006, 0.003\}
- Constant \( \mu_i \) in \{0.008, 0.010\}
- Probabilities for the initial state = \( (1, 0) \) B as starting value
- Probabilities for the initial state = \( (0, 1) \) G as starting value
- Number of simulated scenarios for both the initial states = 1000
- Transition probabilities

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Cumulative Probabilities

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<tbody>
<tr>
<td>Default</td>
<td>0.18</td>
<td>0.16</td>
</tr>
<tr>
<td>Prepayment</td>
<td>0.37</td>
<td>0.39</td>
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The simulations show how the initial CMCs affect the conditional first two moments.

**Conditional Mean Vs Conditional Standard Deviation NPVs**
The difference between the expected NPVs changing the initial state of the CMCs is €128 (~ 8% of the contractual NPV)

The histograms for the NPVs conditional means under different initial CMCs

Contractual NPV = €1,607
NPV Mean
Initial CMCs
Bad = €593 Good = €721
The NPV Standard Deviation is also affected by the CMCs initial state

The histograms for the NPVs conditional standard deviations under different initial CMCs

NPV Standard Deviation
Initial CMCs
Bad = €1,857
Good = €1,647
The variance of the NPV for a single operation is mainly specific

Variance decomposition of the NPV for a single transaction

\[ \text{var}(NPV) = E_H(\text{var}(NPV \mid H)) + \text{var}_H(E(NPV \mid H)) \]

\( E_H(\text{var}(NPV \mid H)) \) is the expected value, under the scenarios’ probability distribution, of the conditional variances (specific component)

\( \text{var}_H(E(NPV \mid H)) \) is the variance, under the scenarios’ probability distribution, of the conditional means (systematic component)

For this example the variance is mainly specific

\[ €3,448,976 = €3,437,527 + €11,448 \] (for initial state = B)
The mean and the variance of a homogeneous portfolio of size $s$ in term of the conditional first two moments and the probability distribution of the $M$ scenarios $q_m$

The NPV mean and variance for a portfolio of $s$ identical transactions

$$PV_s = \sum_{l=1}^{s} NPV_l$$

$$E(PV_s) = E_H(E(PV_s \mid H)) = s \cdot \sum_{m=1}^{M} q_m \cdot E(NPV_1 \mid H_m)$$

$$\text{var}(PV_s) = E_H(\text{var}(PV_s \mid H)) + \text{var}_H(E(PV_s \mid H))$$

$$E_H(\text{var}(PV_s \mid H)) = \sum_{m=1}^{M} q_m \cdot \left[ \left( s \cdot E(NPV_1^2 \mid H_m) \right) + s \cdot (s-1) \cdot E^2(NPV_1 \mid H_m) - \left( s \cdot E(NPV_1 \mid H_m)^2 \right) \right]$$

$$\text{var}_H(E(PV_s \mid H)) = \sum_{m=1}^{M} q_m \cdot (s \cdot E(NPV_1 \mid H_m) - E(PV_s))^2$$
Increasing $s$ the expected NPV and the standard deviation grow, CV is asymptotically strictly positive; the systematic risk component predominates.

**Portfolios NPV of size $s$ (initial state Bad)**

- **With $s = 10,000$**
  - NPV mean = €5.93 mln
  - NPV standard deviation = €1.08 mln
  - CV = 0.183
  - Systematic component of variance = 0.971

- **With $s = 100,000$**
  - NPV mean = €59.3 mln
  - NPV standard Deviation = €10.8 mln
  - CV = 0.180
  - Systematic component of variance = 0.997
NPV is a random variable whose first two moments have to be evaluated taking into account default risk, prepayment risk and CMCs.

CMCs may be described by a homogeneous Markov Chain.

Initial state of the Chain influences NPV first two moments.

NPV for a portfolio of identical loans has a Coefficient of Variation strictly positive and its variance is mainly due to the systematic component (s increasing).
Key references


Quirini L. and L. Vannucci (2010), A New Index of Creditworthiness for Retail Credit Products, *Journal of Operation Research Society*, 61, 455-461


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