Risk-based Pricing, or Price-based Risk?

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Paper In A Nutshell

In a lending relationship

- Creditor: price the debt based on predictions of default risk of Borrower → **risk-based pricing**
- Borrower: the pricing impacts the incentive whether to continue payment (non-default) or not (default) → **price-driven risk**
- Creditor's pricing and default prediction will only be accurate, or "self-fulfilling", if they are compatible with Borrower's incentive

To illustrate this idea,

- Nash Equilibrium is established
- Existence of equilibria and market failure (where no reasonable default predictions and pricing are possible) are discussed
- Results: proposal for simultaneous estimation; testable hypotheses

Looking for opportunities of collaboration on subsequent empirical studies – please contact me (xinxu75@gmail.com) if you are interested!

Setup (1/2): Asset

- Similar to setup of Duffie and Lando (2001), a variation of Leland (1994), Leland and Toft (1996) models
- Fixed probability space $(\Omega, \mathcal{F}, P)$
- Risk-neutral world, with discount rate r
- Asset: ex-dividend value V_t evolves as GBM, with initial value V_0 and dividend yield $\delta > 0$

$$\frac{dV_t}{V_t} = \mu dt + \sigma dZ_t$$

- Tax shields are offered for interest expense, at a tax rate $\theta \in (0,1)$ - Asset owner funds asset by issuing debt to take advantage tax shields
- Asset owner becomes Borrower, and issues a console bond
  - This assumption is not central, term loan may be modelled (see Leland and Toft [1996]) without changing the essence

Setup (2/2): Debt

- Debt amount is D, with constant interest rate c, paid continuously
- Default mechanism: Borrower will default and file bankruptcy when asset value V_t first touches a lower bound, V_B → "default boundary"
  - Similar to Black and Cox (1976) model
  - The (term structure) of PD is the first passage probability from V_0 to V_B
- After default: Creditor will get a fraction γ of the present value (PV) of future cash flows; Borrower will get nothing
  - PV of cash flows: $\frac{\delta V_B}{r-\mu}$
  - Loss of $(1-\gamma)$ is due to the cost of market frictions, understood as "haircuts" in LGD calculation. γ may be interpreted as recoverability of asset

Creditor determines pricing c; Borrower determines default boundary/risk V_B

All other parameters are assumed to be exogenous and publicly known

- Exogeneity of D is a simplistic assumption, and will be discussed later

Problem for Creditor

- Let τ denote the first passage time from V_0 to V_B (i.e., the random default time) and let $\mathrm{F}(\tau \leq y)$ and $f(\tau)$ denote CDF and PDF of τ, respectively
- Creditor is to find c given Borrower's default risk and the following condition: *If debt were to be sold immediately after origination, there would be no gain, no loss*

$$D = \underbrace{\underbrace{\int_0^{\infty} e^{-ry} cD[1 - \mathrm{F}(\tau \leq y)]\, dy}_{\text{PV of cash flows if non-default}} + \underbrace{\int_0^{\infty} e^{-r\tau} \gamma \frac{\delta V_B}{r - \mu} f(\tau)\, d\tau}_{\text{PV of cash flows if default}}}_{\text{Market value of the debt}}$$

D: Creditor's investment

PV of cash flows if non-default

PV of cash flows if default

Market value of the debt

Solution for Creditor

- $F(\tau \leq y)$ and $f(\tau)$ are well known in this setup (see Harrison [1990]), and the above equation becomes

$$D = \frac{cD}{r}\left[1 - \left(\frac{V_0}{V_B}\right)^{-\alpha}\right] + \gamma \frac{\delta V_B}{r-\mu}\left(\frac{V_0}{V_B}\right)^{-\alpha},$$

where $\alpha = \frac{\left(\mu - \frac{\sigma^2}{2}\right) + \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}}{\sigma^2}$.

- Re-arranging it, one obtains the solution for Creditor

$$c^*(V_B) = \frac{D - \gamma \frac{\delta V_B}{r-\mu}\left(\frac{V_0}{V_B}\right)^{-\alpha}}{\frac{D}{r}\left[1 - \left(\frac{V_0}{V_B}\right)^{-\alpha}\right]}.$$

Problem for Borrower

- Optimal default policy framework from Leland & Toft (1996), a classic optimal control problem
- Borrower is to find V_B given Creditor's pricing and the following optimization problem:

$$\operatorname*{Max}_{V_B} \quad Q_0 + D$$

$$s.t.\, Q_t \geq 0 \text{ (limited liability)}, \; V_B > 0, \; \forall t,$$

where $Q_t = \mathrm{E}\left[\int_t^\tau e^{-r(s-t)}(\delta V_s - (1-\theta)cD)\,\mathrm{d}s \middle| \mathcal{F}_t\right]$ is the equity value at time t, i.e. the expected PV of after-tax net cash flows prior to default

- Note that Borrower maximize the firm value (equity + debt), so there is no conflict of interest between Creditor and Borrower in our model
- D was provided earlier, Q_0 is given next. It turns out that $(Q_0 + D)$ is monotonically decreasing in V_B, thus the optimal V_B^* is obtained from the binding constraints

Solution for Borrower

- Conjecture the value function $J(V_t, t) = \underset{V_B}{\text{Max}}\, Q_t$, which satisfies Hamilton-Jacobi-Bellman (HJB) Equation

$$0 = \underset{V_B}{\text{Max}}\left[\frac{\partial J}{\partial t} + \mathcal{L}J - rJ + (\delta V_t - (1-\theta)cD)\right], V_t > V_B,$$

where $\mathcal{L}J$ is the differential generator of J, $\mathcal{L}J = \mu V_t \frac{\partial J}{\partial V_t} + \frac{\sigma^2}{2} V_t^2 \frac{\partial^2 J}{\partial {V_t}^2}$.

- Assumption of time-homogeneity (i.e., J is not a function of time t): PDE becomes ODE
- Boundary condition: $J(V_B^*) = 0, V_t \le V_B^*$
- “Smooth-pasting” condition: $\frac{dJ}{dV_t}|_{V_t = V_B^*} = 0$
- Verification Theorem (Duffie and Lando [2001])

Solution is

$$J(V_t) = \begin{cases} \frac{\delta V_t}{r-\mu} - \frac{\delta V_B}{r-\mu}\left(\frac{V_t}{V_B}\right)^{-\alpha} - (1-\theta)\frac{cD}{r}\left[1 - \left(\frac{V_t}{V_B}\right)^{-\alpha}\right], V_t > V_B; \\ 0, V_t \le V_B \end{cases}$$

$$Q_0 = J(V_0);$$

Note that $(Q_0 + D)$ is decreasing in V_B. The constraint-binding V_B that maximizes equity is obtained using the smooth-pasting condition,

$$V_B^*(c) = \frac{(1-\theta)\alpha}{(1+\alpha)} \frac{(r-\mu)}{\delta} \frac{cD}{r} > 0$$

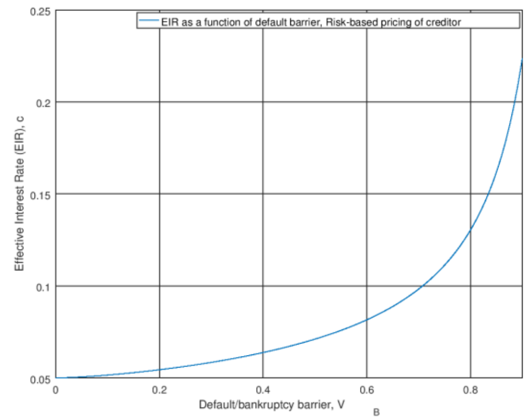
Nash Equilibrium

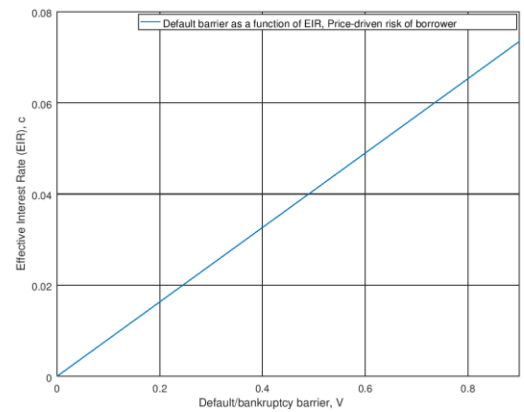
- Creditor:

$$c^*(V_B) = \frac{D - \gamma \frac{\delta V_B}{r-\mu}\left(\frac{V_0}{V_B}\right)^{-\alpha}}{\frac{D}{r}\left[1 - \left(\frac{V_0}{V_B}\right)^{-\alpha}\right]},$$

- Borrower:

$$V_B^*(c) = \frac{(1-\theta)\alpha}{(1+\alpha)} \frac{(r-\mu)}{\delta} \frac{cD}{r} > 0.$$





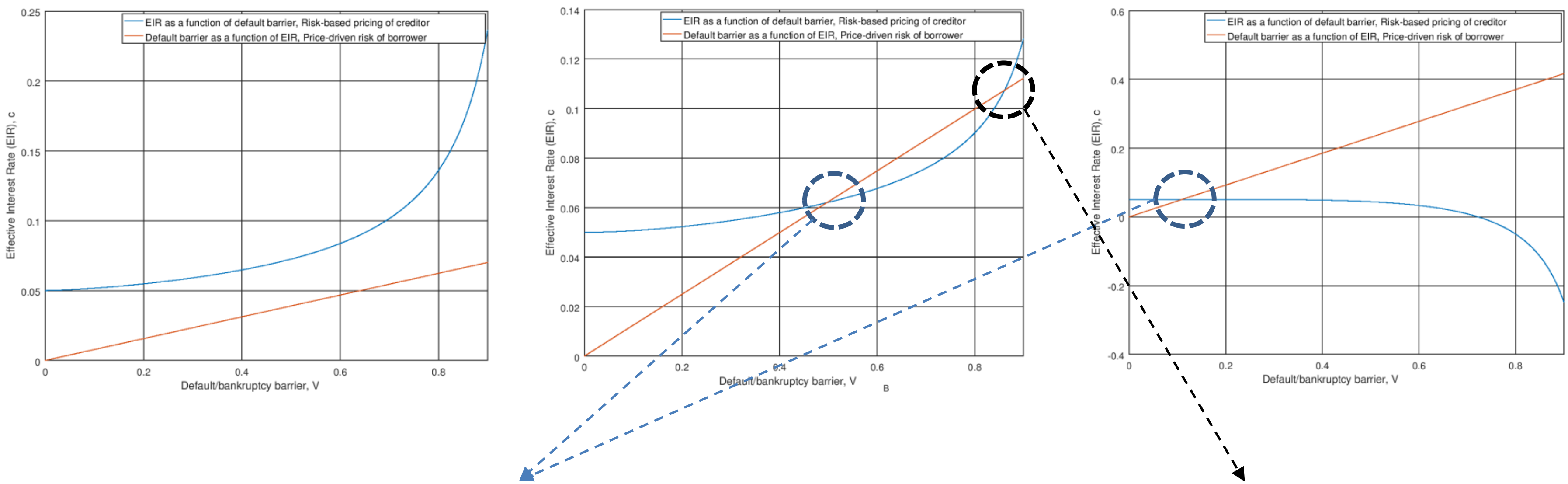
- In equilibrium, c^* and V_B^* are simultaneously determined

Equilibria

Solutions may not be unique, or even not exist

- Without Loss of Generality, assuming initial asset book value $V_0 = 1$
- Example: changing ex-dividend expected return of asset, μ, and fixing all other parameters at moderate values

$\mu = 1.5\%$ No Equilibria | $\mu = 3\%$ Two Equilibria | $\mu = 4.5\%$ One Equilibria



"Stable" equilibria, behaving intuitively and with reasonable "credit spread" $(c^* - r)$

"Unstable" equilibria, behaving weirdly

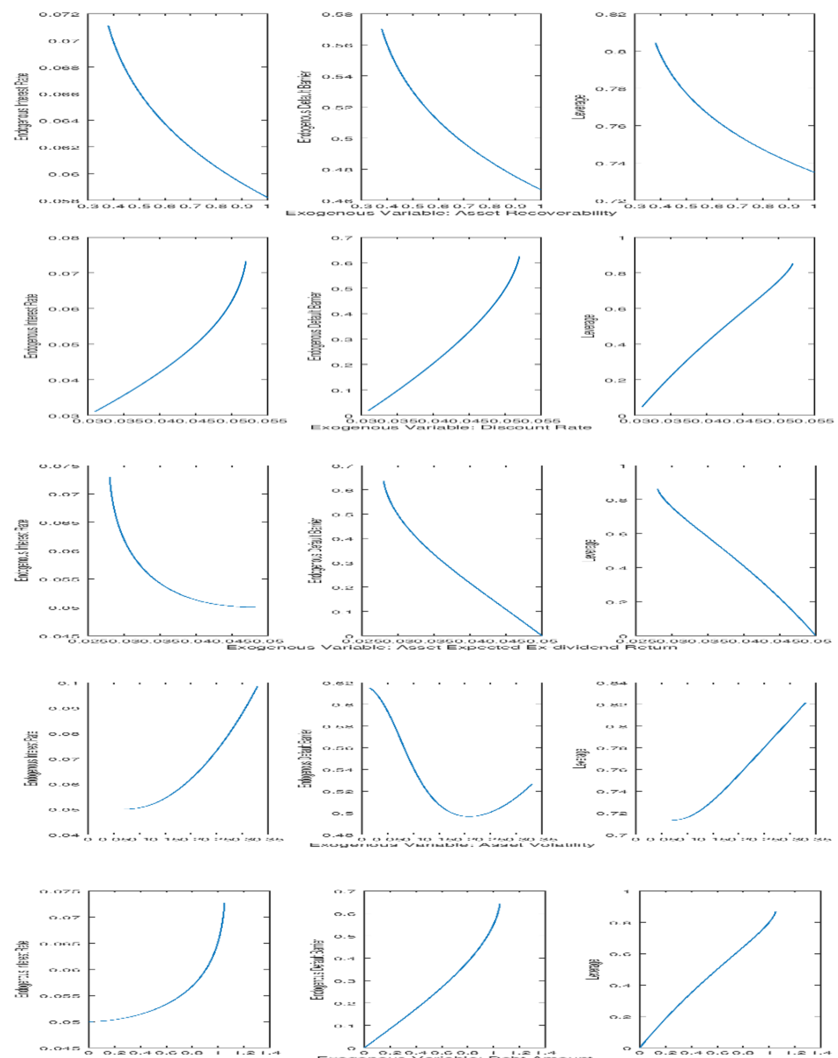
Non-existence of Equilibrium, Market Failure

- If equilibrium exist, Creditor will accurately predict Borrower's default risk, and Borrower has the incentive to honor this prediction. There will be no "surprise"
- If equilibrium do not exist, debt should not be issued. Otherwise, no matter how Creditor predicts the default risk, the prediction will under-estimate the actual risk determined by Borrower, incurring a loss

	$\mu = 3\%$ **(equilibrium exists)**	$\mu = 1.5\%$ **(no equilibrium)**
Debt amount	0.95	0.95
Creditor's prediction of Borrower's default barrier V_B	0.4968	0.55
Creditor's pricing based on default risk prediction	6.196% (discount rate *r*=5%)	7.7537% (*r*=5%)
Given pricing, Borrower's optimal $V_B^*(c)$	0.4968	0.99486
The market value of the debt based on V_B^*	0.95	0.406 (loss=0.95-0.406)

Comparative Statics within Feasible Regions

Exogenous Variables	c^*	V_B^*	**Leverage** $\left(\frac{D}{Q_0+D}\right)$
Asset recoverability γ	↓	↓	↓
Discount rate r	↑	↑	↑
Asset expected ex-dividend return μ	↓	↓	↓
Asset volatility σ	↑	↓↑	↑
Debt amount D	↑	↑	↑



Empirical Implications

Simultaneous Equation Modelling (SEM) approach for estimating default risk and pricing consistently and robustly

- Credit spreads or EIR as an endogenous variable to predict default risk
- GMM, IV (2SLS) or other SEM estimation are required, in which typically instrument variable(s) need to be identified
- Reduced-form default risk and pricing models may be adopted, instead of structural models (in this presentation)

Relaxing some assumptions in empirical studies

- Borrower’s liquidity may be considered as an exogenous variable – default may be caused by either optimal default barrier or liquidity constraint
- Debt amount *D* may be another endogenous variable, determined by other parameters
  - One needs to model mechanism of debt amount determination
  - There may be three simultaneous equations in this system

Empirically Testable Hypotheses

Tests during "crisis" periods and "normal" periods

- "Crisis" periods: Why does non-existence of equilibrium occur? Could debts still be issued under these circumstances in reality?
  - Asset-related parameters
    - Recoverability is too low (e.g. unsecured debts, or lower seniority)
    - Expected return is too low
    - Volatility is too high
  - Information asymmetry or incomplete information (about above asset-related parameters)
  - Discount rate, or market expected return, is too high (or trivially too low)
  - Debt over-burden
- "Normal" periods: If equilibrium exist, comparative statics may be tested

Please contact me if you are interested in the empirical proposal. I am open to ideas of collaboration in terms of research, funding, data…

Acknowledgement

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